Finite Math - Spring 2017

Lecture Notes - 2/8/2017

Homework

- Section 2.5 - 59, 60, 62
- Section 2.6 - 13, 16, 18, 20, 27, 30, 32, 43, 44, 47

Section 2.5 - Exponential Functions

A Special Number: e. There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which the quantity

$$\left(1 + \frac{1}{x}\right)^x$$

approaches as x tends towards ∞.

This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Example 1. *In 2013, the estimated world population was* 7.1 *billion people with a relative growth rate of* 1.1%*.*

(a) *Write a function modeling the world population t years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Solution.

(a) *We will write our function to output in billions. We will treat $t = 0$ as the year 2013, so that the initial population for this model is $c = 7.1$. The relative growth rate is 1.1%, which we must convert to a decimal before using $r = 0.011$. t will measure the years since 2013. Plugging these into the model, we get*

$$\textit{Population} = P = 7.1e^{0.011t} \text{ \textit{billion}.}$$

(b) *To find the estimated population in these years, we just need to plug the appropriate t-value into the model above.*

2015) $t = 2$

$$P = 7.1e^{0.011(2)} = 7.1e^{0.022} \approx 7.26 \textit{ billion}$$

2025) $t = 12$

$$P = 7.1e^{0.011(12)} = 7.1e^{0.132} \approx 8.1 \textit{ billion}$$

2035) $t = 22$

$$P = 7.1e^{0.011(22)} = 7.1e^{0.242} \approx 9.04 \textit{ billion}$$

Example 2. *The population of some countries has a relative growth rate of* 3% *per year. Suppose the population of such a country in 2012 is* 6.6 *million.*

(a) *Write a function modeling the population t years after 2012.*

(b) *What is the expected population in 2017? 2020?*

Solution.

(a) $P = 6.6e^{0.03t}$

(b) 7.67 *million;* 8.39 *million*

Section 2.6 - Logarithmic Functions

Before we can accurately talk about what logarithms are, let's first remind ourselves about inverse functions.

Inverse Functions. The inverse of a function is given by running the function backwards. But when can we do this?

Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?

Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

This shows that not every function is invertible. To get the inverse of a function, we need each range value to come from *exactly one* domain value. We call such functions *one-to-one*.

If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms. We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition 1 (Logarithm)**.** *The* logarithm of base b *is defined as the inverse of* b^x*. That is,*

$$y = b^x \Longleftrightarrow x = \log_b y.$$

Since the domain and range switch when we take inverses, we have

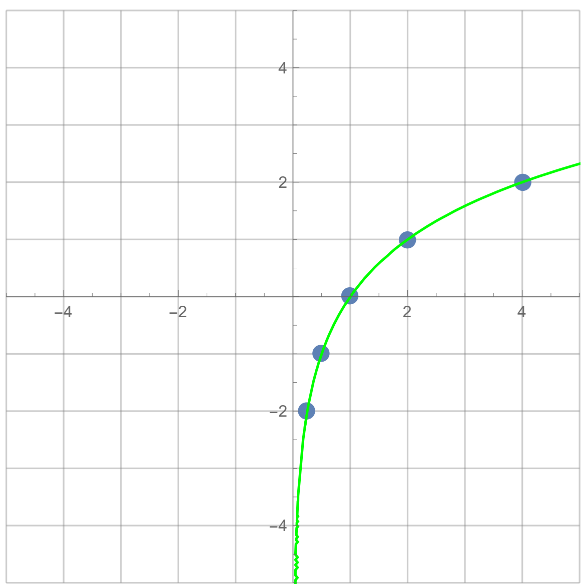
function	domain	range
$f(x) = b^x$	$(-\infty, \infty)$	$(0, \infty)$
$f(x) = \log_b x$	$(0, \infty)$	$(-\infty, \infty)$

Let's look at one example of a graph of a logarithmic function.

Example 3. *Sketch the graph of* $f(x) = \log_2 x$*.*

Solution. *To get the points for this, we can just recognize that it is the inverse of* 2^x *so we take each of those points and flip the* x *and* y *coordinates. This gives*

x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$f(x)$	-2	-1	0	1	2



Properties of Logarithms. Since logarithms are inverse to exponential functions, we get some convenient properties for logarithms:

Property 1 (Properties of Logarithms)**.** *Let* $b, M, N > 0$*,* $b \neq 1$*, and* p, x *be real numbers. Then*

(1) $\log_b 1 = 0$

(2) $\log_b b = 1$

(3) $\log_b b^x = x$

(4) $b^{\log_b x} = x$

(5) $\log_b MN = \log_b M + \log_b N$

(6) $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

(7) $\log_b M^p = p \log_b M$

(8) $\log_b M = \log_b N$ *if and only if* $M = N$

Properties 3 and 7 above are incredibly important to us as we will use them frequently in the study of financial mathematics! Learn these properties well!!

The Natural Logarithm. Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm, the *natural logarithm*.

$$\log_e x = \ln x.$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Logarithms and Exponents.

Example 4. *Solve for x in the following equations:*

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x - 4)^3$

Solution.

(a) *Begin by dividing both sides by* 2 *to get*

$$3.5 = e^{0.2x}.$$

Next, apply ln *to both sides to get*

$$\ln 7 = \ln e^{0.2x} = 0.2x$$

so that dividing by 0.2 *gives*

$$x = \frac{\ln 3.5}{0.2} \approx 6.26781.$$

(b) *Straight away, we apply* ln *here to get*

$$\ln 16 = \ln 5^{3x} = 3x \ln 5.$$

Solving for x here gives us

$$x = \frac{\ln 16}{2 \ln 5} \approx 0.86135.$$

(c) *In this example, we have to begin by taking the cube root of both sides, that is, raising both sides to the $\frac{1}{3}$ power:*

$$8000^{\frac{1}{3}} = \left((x-4)^3\right)^{\frac{1}{3}} = x - 4.$$

Then, solving for x gives:

$$x = 8000^{\frac{1}{3}} + 4 = 20 + 4 = 24.$$

A quick reminder of different types of exponents:

- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{1/2} = \sqrt{a}$
  - $a^{1/3} = \sqrt[3]{a}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$